Thermocapillary motion of a drop in a fluid under external gradients.
Faxén theorem

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Abstract

We have studied the thermocapillary motion of a drop in a fluid under nonequilibrium conditions, in the presence of velocity and temperature gradients. After reformulating the boundary value problem in terms of an induced force and an induced heat source densities, we have derived the Faxén theorem for the drop. The theorem gives the hydrodynamic force exerted on the drop as a function of its velocity and of the unperturbed velocity and temperature fields. From it we infer expressions for the mobility and thermocapillary coefficients. Our general result then permits us to analyze a number of particular situations of interest which have been reported by other authors.

1. Introduction

In the study of the dynamics of a particle immersed in a carrier fluid, one is usually concerned with the calculation of the force exerted by the fluid on the particle. The problem has been formulated and solved focusing on different situations of interest. A classical and vast literature on the subject exists [1,2]. Furthermore, the knowledge of the force acting on the particle in terms of the unperturbed hydrodynamic fields, as given by the Faxén theorem [3], constitutes the starting point in the study of the Brownian motion, since it allows us to derive the Langevin equation as well as the fluctuation-dissipation theorem [4].

In this paper, we are particularly interested in the stationary motion of drops or bubbles through a fluid subjected to external velocity and temperature gradients. In particular,
our main purpose will be the formulation of the Faxén theorem when both gradients are present. The motion of drops is sensitive to the processes taking place at their interface. Even in the absence of interfacial dissipative effects, the surface tension may induce interesting effects in the movement of a drop. In particular, when the interfacial tension depends only on temperature, it gives rise to thermocapillary effects, which has implications on the Faxén theorem.

Up to now, only partial solutions of the problem have been reported in the literature. Thus, if the fluid in the absence of the drop is in stationary motion and in the absence of capillary effects, an expression for the Faxén theorem was derived in Ref. [5]. The thermocapillary induced motion of a bubble was first investigated by Young et al. [6] who derived an expression for the rising velocity in a vertical temperature gradient. Subramanian [7] derived the expression for the force exerted on a drop in terms of the unperturbed temperature gradient. Harper et al. [8] introduced the heat interchange between the interface and bulk liquids, which originates from the motion of the drop, and applied it to the study of the mobility of a bubble in the absence of external temperature gradients. More recently, Torres et al. [9] have studied the movement of drops and bubbles in a fluid at rest and the temperature gradient generated by their motion, estimating also the importance of this phenomenon for a wide variety of fluids. Though usually neglected, this effect may be of importance and has implications in the motion of drops and bubbles in thermal equilibrium or subject to thermal gradients.

In the derivation of the Faxén theorem one may adopt different methods. The most usual one consists on its derivation making use of the reciprocal relations [2]. Alternatively, it can be obtained by introducing induced field densities [10]. Concerning the motion of drops, results obtained using the first procedure are reported in Ref. [11], whereas the second one has been applied to the study of the mobilities of a suspension of drops in the absence of thermocapillary effects [12].

As we will see throughout this paper, the relative velocity of the drop with respect to the external fluid is the sum of two contributions. One is proportional to the external force, whereas the other one is proportional to the external temperature gradient, the proportionality coefficients being the mobility and the thermocapillary coefficient, respectively. The presence of a temperature gradient may induce motion of the drop, referred to as thermocapillary motion, and will cause a coupling in the different contributions to the motion of the drop, since the mobility and thermocapillary coefficients of the drop depend on the inhomogeneities of the surface tension which cause the thermocapillary effect.

This paper is organized in the following way: in Section 2 we formulate the boundary value problem in which the surface of the drop defines two domains with different transport properties. The presence of the drop is then substituted by introducing appropriate induced force and heat source densities. We then obtain the formal solution for the velocity and temperature fields. Section 3 is devoted to the derivation of the Faxén theorem. The resulting Faxén theorem is given in Section 3.3 and it expresses the force exerted on the drop in terms of the velocity of its center of mass, the velocity of the fluid in the absence of the drop, and the externally imposed temperature gradient. If
one is not interested in the details of the derivation, it is possible to continue reading Section 3.3 and the concluding section immediately after the introduction. From our general expression we consider a number of particular cases of interest which have been analyzed previously. Finally, in Section 4 we point out our main results.

2. Formulation of the problem

Here we will consider the stationary movement of a drop immersed in a fluid which itself is subject to conditions creating velocity and temperature gradients. Both the fluid inside the drop and the surrounding fluid are assumed to be incompressible and Newtonian, and they are characterized by viscosities $\eta^i$ and $\eta^o$ and thermal conductivities $\lambda^i$ and $\lambda^o$, respectively. The interface between both fluids has a surface tension, $\gamma$, which, in general, may depend on the temperature. Since we are dealing with pure liquids, no mass is adsorbed at the interface, and, for the sake of simplicity, we have disregarded surface dissipative effects. The pressure tensor at any point of the space is then given by [13]

$$P = \Theta P_- + (1 - \Theta) P_+ + \delta_s P_S$$

with $\Theta(r)$ being a Heaviside step function which has value 1 for points inside the drop and 0 outside. From now on, the subscripts $+, -$ will refer to the value of the corresponding field in the outer and inner fluids, respectively, and the subscript $S$ refers to the value of the corresponding surface excess quantity. Moreover, the surface delta function $\delta_s$ is defined by

$$\nabla \Theta = -\hat{n} \delta_s,$$

with $\hat{n}$ the unit vector normal to the interface and pointing towards the outer fluid. Explicit expressions for the pressure tensors in each phase and at the interface are

$$P_\pm = p_\pm \mathbf{1} - \eta^{\alpha \beta} (\nabla v_\pm + (\nabla v_\pm)^T),$$

$$P_S = -(1 - \hat{n} \hat{n}) \gamma,$$

where $\mathbf{1}$ is the unit tensor and the superscript $T$ means the transpose of the corresponding tensor. Note that in Eq. (2.4) no dissipative contribution to the surface excess pressure tensor has been considered.

We assume that both the Reynolds and Péclet numbers are small enough in order to use the linearized Navier-Stokes and Fourier equations. Moreover, the capillary number $\tau_0 v / \gamma$, with $v$ a typical velocity on the surface of the drop, is sufficiently small to ensure that the shape of the drop remains approximately spherical, with radius $a$ [14]. Therefore, we are restricting our study to linear deviations from equilibrium, imposed by the external conditions. We will consider nonuniform velocity and temperature distributions that, in the absence of the drop, are denoted as $v_0(r)$ and $T_0(r)$, and an external force $K^{\text{ext}}$ uniformly distributed in the drop with density $F^{\text{ext}} = K^{\text{ext}} / (\frac{4}{3} \pi a^3)$. For example, gravity provides such a force distribution.
With these assumptions, the velocity field must satisfy the stationary linearized equation
\[ \nabla \cdot P = F^{\text{ext}}. \] (2.5)

Then, choosing the center of the coordinate system at the center of the drop, one has for the different phases
\[ \nabla p - \eta \Delta v = 0, \quad r > a; \] (2.6)
\[ \nabla p - \eta \Delta v = F^{\text{ext}}, \quad r < a; \] (2.7)
\[ (P_+ - P_-) \cdot \mathbf{n} = \nabla_s \gamma + 2H \gamma \mathbf{n}, \quad r = a. \] (2.8)

Eqs. (2.6) and (2.7) are the stationary Navier-Stokes equations corresponding to the outer and inner fluids, respectively. The last equation (2.8) is a boundary condition relating the discontinuity of the pressure tensor through the interface with the surface tension and its gradients. In this equation we have introduced the surface gradient, defined as the surface projection of the gradient operator \[ \nabla_s = (1 - \mathbf{n} \cdot \nabla) \cdot \nabla. \] (2.9)

Furthermore, \( H = -\nabla_s \cdot \mathbf{n} \) is the mean curvature of the interface, being \(-1/a\) for a spherical surface of radius \( a \). It should be noted that small deviations from this value appear as a consequence of small deformations of the drop caused by its motion. However, these deviations will appear as second order contributions in the force acting on the drop and will be neglected in the derivation of the Faxén theorem.

In Eq. (2.8), the term proportional to the gradient of the surface tension along the interface couples the velocity to the temperature field, and its presence will be the origin of a thermocapillary force. This boundary condition has to be supplemented by the continuity of the velocity field across the interface
\[ v_+ = v_-, \quad r = a; \] (2.10)
and the boundary condition of the velocity far from the drop
\[ v \to v_0(r), \quad r \to \infty; \] (2.11)
ensuring that the velocity field tends to the unperturbed velocity profile \( v_0(r) \) far from the drop. Moreover, we should introduce the center of mass velocity of the drop, \( \mathbf{u} \), as
\[ \mathbf{u} = \frac{3a^3}{4\pi} \int v(r) \, dr, \] (2.12)
where \( V \) denotes the volume of the drop. It is interesting to note that in Eq. (2.10) we have not assumed, as it is usually done, that the normal component of the velocity field at the surface coincides with the normal component of the center of mass velocity of the drop \( \mathbf{n} \cdot v = \mathbf{n} \cdot \mathbf{u} \). Since we do not need to solve for the velocity field, the use of
this boundary condition can be avoided. Finally, we will consider incompressible fluids and, consequently, for both phases we have the condition

$$\nabla \cdot \nu = 0. \quad (2.13)$$

As we consider a temperature-dependent surface tension, the velocity field will be coupled with the temperature field, as shown in Eq. (2.8). Therefore, in order to determine the velocity profile we should study the temperature field simultaneously. For small Péclet numbers, the heat flux is given by the Fourier law valid in each of the two bulk phases, and can be written in the compact form

$$\mathbf{J} = -\Theta \lambda' \nabla T - (1 - \Theta) \lambda \nabla T,$$  

where we have neglected the heat current along the interface.

To formulate the boundary value problem for the temperature field we must realize that a heat source exists which originates from the motion of the drop. In fact, when the particle moves, the incident fluid creates a distribution of stresses at the surface of the drop causing local compressions and expansions along the interface. The heat rate per unit area generated during the process, $j_s$, is then given by [8,13,9]

$$j_s = (e_s - \gamma) \nabla_s \cdot \nu,$$  

where $e_s$ is the surface excess internal energy. For pure fluids, $e_s$ is approximately constant and one can show the relation [8]

$$e_s - \gamma = -T \frac{d\gamma}{dT}, \quad (2.16)$$

where $d\gamma/dT \equiv \gamma_T$ is also approximately constant.

The internal energy balance equation is then given by

$$\nabla \cdot \mathbf{J} = -j_s \delta_s, \quad (2.17)$$

which implies

$$\nabla T = 0, \quad (r \neq a) \quad (2.18)$$

$$\mathbf{n} \cdot (\mathbf{J}_- - \mathbf{J}_+) = -(e_s - \gamma) \nabla_s \cdot \nu, \quad r = a. \quad (2.19)$$

The heat flux, therefore, has a discontinuity across the interface related to the heat exchange due to the thermocapillary effect. It is also interesting to emphasize that because of this boundary condition, the temperature field is also coupled to the velocity field. Finally, we should consider that the temperature field is continuous across the interface, and that far from the drop it approaches the unperturbed temperature field

$$T_+ = T_-; \quad r = a; \quad (2.20)$$

$$T(r) \to T_o(r), \quad r \to \infty. \quad (2.21)$$
2.1. The induced density fields

The boundary value problem formulated through equations Eqs. (2.6)-(2.8), (2.10)-(2.13) and (2.18)-(2.21) can be solved by means of an extension of the induced force method [10,16]. This procedure consists of the formulation of an equivalent problem in which a homogeneous fluid with the same physical properties as the outer fluid moves under the action of a distribution of forces, \( F_{\text{ind}} \), and a distribution of heat sources, \( Q_{\text{ind}} \). These induced fields are defined in such a way that the velocity and temperature fields are the same as those of the original problem and that the appropriate boundary conditions are fulfilled.

The previous boundary value problem is then reformulated as

\[
\begin{align*}
\nabla p' - \eta^0 \triangle v &= \alpha^{-1} F_{\text{ext}} + F_{\text{ind}}, \\
-\lambda' \triangle T &= Q_{\text{ind}},
\end{align*}
\]

which are now valid in the whole space, along with the boundary conditions ensuring the appropriate asymptotic behavior of the fields far from the drop, specified in Eqs. (2.11) and (2.21). In Eq. (2.22) we have introduced the ratio \( \alpha = \eta'/\eta^0 \).

It has been shown [12] that in the stationary situation the induced force is zero for \( r < a \) if one chooses the new pressure field \( p' \)

\[
p' = \begin{cases} 
  p, & r > a; \\
  \alpha^{-1} p, & r < a;
\end{cases}
\]

Then, the induced force and heat source densities are different from zero only at the interface, and they can be expressed in the more appropriate form

\[
\begin{align*}
F_{\text{ind}}(r) &= \delta_s(r) f_{\text{ind}}(r), \\
Q_{\text{ind}}(r) &= \delta_s(r) q_{\text{ind}}(r),
\end{align*}
\]

where the surface densities, \( f_{\text{ind}}(r) \) and \( q_{\text{ind}}(r) \), are related to the discontinuities in the derivatives of the velocity and temperature fields. In fact, using the boundary conditions given in Eqs. (2.8) and (2.19) we can express the induced fields as a function of the velocity and temperature fields at the interface and in one of the bulk phases, namely

\[
\begin{align*}
f_{\text{ind}}(r) &= (1 - \alpha^{-1}) \hat{n} \cdot P_+ - \alpha^{-1} \nabla_s \cdot P_s, \\
q_{\text{ind}}(r) &= (1 - \beta^{-1}) \hat{n} \cdot J_+ - \beta^{-1} (e_s - \gamma) \nabla_s \cdot v,
\end{align*}
\]

where \( \beta = \lambda'/\lambda^0 \).

In our subsequent analysis we will need the formal solution for the temperature and velocity fields. From Eq. (2.22) and the incompressibility condition, Eq. (2.13), we obtain

\[
\nu(r) = \nu_0(r) + \int_{r' \leq a} dr' \mathbf{H}(r - r') \cdot \left[ \alpha^{-1} F_{\text{ext}}(r') + F_{\text{ind}}(r') \right],
\]
with \( H(r - r') \) being the Oseen tensor

\[
H(r) = \frac{1}{8\pi \eta r} \left( 1 + \frac{rr}{r^2} \right). \tag{2.30}
\]

In a similar way, we can derive from Eq. (2.23) the formal solution for the temperature field

\[
T(r) = T_0(r) + \int_{r' \leq a} dr' G(r - r') Q_{\text{ind}}(r'), \tag{2.31}
\]

where we have introduced the Green function

\[
G(r) = \frac{1}{4\pi \lambda'^2 r}. \tag{2.32}
\]

The formal solutions given in Eqs. (2.29) and (2.31), together with the expressions for the induced fields (2.27) and (2.28), form a closed set of equations, equivalent to the original boundary value problem formulated through Eqs. (2.6)-(2.8), (2.10)-(2.13) and (2.18)-(2.21), from which any property of the velocity and temperature fields can be computed.

3. Derivation of the Faxén theorem

Our purpose in this section is to derive the Faxén theorem, which expresses the force exerted by the surrounding fluid on the drop in terms of the drop velocity and surface averages of the unperturbed hydrodynamic fields. Such a force is defined as

\[
K^{\text{hyd}} = - \int_S dS \hat{n} \cdot \mathbf{P}_t. \tag{3.1}
\]

The advantage of the introduction of the induced field densities lies in the fact that it will allow us to obtain a Faxén theorem through the calculation of certain averages of the hydrodynamic fields, without having to know their explicit values in the whole space [10]. By using the formal solutions Eqs. (2.29) and (2.31) we will be able to arrive at a relationship between the velocity of the drop and the external force and the unperturbed hydrodynamic fields. Due to the linearity of the problem, this relationship must be of the form

\[
\mathbf{u} = \mathbf{v}_0 + \mu K^{\text{ext}} + \tau \nabla T_0, \tag{3.2}
\]

where \( \mu \) is the mobility of the drop and \( \tau \) will be called the thermocapillary coefficient.

The translational velocity of the drop is defined as the average of the velocity field in the volume of the drop. The volume average will be noted as

\[
\ldots V \equiv \frac{1}{\frac{4}{3}\pi a^3} \int_{V} \ldots d\mathbf{r}. \tag{3.3}
\]
Therefore, if we consider the volume average of the formal expression of the velocity field (2.29), and using the average of the Oseen tensor computed in Appendix A, (A.7), we obtain

\[ u = \bar{v}_0^V + \int_{r \leq a} dr' \bar{H}(r - r')^V \cdot \left[ \alpha^{-1} F^{\text{ext}}(r') + F^{\text{ind}}(r') \right] \]

\[ = \bar{v}_0^V + \frac{a}{5\eta^o} (3I + \hat{n}\hat{n}) \cdot \mathbf{f}^{\text{ind}} + \frac{K^{\text{ext}}}{5\pi\eta^o a\alpha} , \]

(3.4)

where we have also introduced the surface average

\[ \bar{\int}_S \equiv \frac{1}{4\pi a^2} \int_S \cdots dS. \]

(3.5)

Here \( S \) refers to the surface of the drop. Eq. (3.4) shows that the velocity of the drop depends on the external force, the unperturbed velocity field, and the surface average of some components of the induced force distribution. Therefore, in order to arrive at an expression of the form (3.2), we have to relate this average to the external force and gradients.

To this end, we will use the decomposition \((3I + \hat{n}\hat{n}) \cdot \mathbf{f}^{\text{ind}} = 4\mathbf{f}^{\text{ind}} - (1 - \hat{n}\hat{n}) \cdot \mathbf{f}^{\text{ind}}\) and compute the average of each contribution separately. The surface average of the induced force can be obtained from Eq. (2.27)

\[ \mathbf{f}^{\text{ind}}_S = (1 - \alpha^{-1}) \hat{n} \cdot \mathbf{P}^S_+ - \alpha^{-1} \nabla_s \cdot \mathbf{P}^S_s . \]

(3.6)

The last term on the right hand side of Eq. (3.6) vanishes, since it is the integral of a surface divergence over a closed surface [18]. The first term is proportional to the total hydrodynamic force acting on the drop, \( K^{\text{hyd}} \), that must be balanced with the total external force as can be seen inserting the boundary condition Eq. (2.8) in Eq. (3.1) and using the momentum balance (2.5)

\[ K^{\text{hyd}} = \int_S dS \left[ -\hat{n} \cdot \mathbf{P}_- + \nabla_s \cdot \mathbf{P}_s \right] = -K^{\text{ext}}. \]

(3.7)

Therefore, from Eqs. (3.6) and (3.7) we arrive at

\[ \mathbf{f}^{\text{ind}}_S = \frac{1 - \alpha^{-1}}{4\pi a^2} K^{\text{ext}}. \]

(3.8)

The surface average of the induced force is, then, proportional to the total external force acting on the drop. This result agrees with the corresponding expression obtained in Ref. [12] in the absence of thermocapillary effects although in that reference the factor \( 1 - \alpha \) is not present because the induced force is defined in a slightly different form. It should be noted that this difference is a matter of choice and does not affect the force on the drop.

Regarding the surface projection of the induced force, we multiply Eq. (2.27) by the surface projector, \((1 - \hat{n}\hat{n})\), and after taking the surface average we get
(1 - \hat{n} \cdot \mathbf{P}_+ \cdot (1 - \hat{n} \hat{n})) \cdot f_{\text{ind}}^S = (1 - \alpha^{-1}) \hat{n} \cdot \mathbf{P}_+ \cdot (1 - \hat{n} \hat{n})^S - \alpha^{-1} (1 - \hat{n} \hat{n}) \cdot (\nabla_s \cdot \mathbf{P}_s)^S. \\
(3.9)

The first term on the right hand side relates the average of the induced force to the gradients of the velocity field. This term would exist even if the surface tension were constant. The second term depends on the gradient of the surface tension, which is assumed to depend only on the temperature; therefore, it relates the induced force to the temperature gradients. In the next subsections we will compute both averages in detail.

3.1. The velocity field averages

In order to calculate the average of \( \hat{n} \cdot \mathbf{P}_+ \cdot (1 - \hat{n} \hat{n}) \), we will use the constitutive relation (2.3) to write it as a function of the velocity gradients,

\[
\hat{n} \cdot \mathbf{P} \cdot (1 - \hat{n} \hat{n}) = -\eta \frac{a}{3} ((\nabla \nu) \cdot \hat{n} + (1 - 2 \hat{n} \hat{n}) \cdot (\hat{n} \cdot \nabla) \nu). \\
(3.10)
\]

The surface averages of the velocity field have already been calculated in Ref. [17]. One gets

\[
\frac{\partial}{\partial r} [\hat{\nu}^{S(r)} - \frac{2}{3} \eta \frac{a}{3} ((\nabla \nu) \cdot \hat{n} + (1 - 2 \hat{n} \hat{n}) \cdot (\hat{n} \cdot \nabla) \nu)] \\
= \frac{2 \eta}{a} [\hat{\nu}^{S(r)} - \frac{a}{2} \frac{\partial}{\partial r} \hat{\nu}^{S(r)}]_{r=a^+}. \\
(3.11)
\]

To arrive at an explicit expression in terms of the unperturbed fields, we should take into account that the volume average is equal to the translational velocity of the drop. Moreover, the surface averages are calculated using the formal solution Eq. (2.29). One then obtains

\[
\hat{\nu}^{S} = \hat{\nu}^{S} + \int_{r \leq a} dr' \mathbf{H}(r - r') \cdot [\alpha^{-1} \mathbf{F}^{\text{ext}}(r') + \mathbf{F}^{\text{ind}}(r')] \\
= \hat{\nu}^{S} + \frac{1}{6\pi \eta a} K^{\text{ext}}, \\
(3.12)
\]

\[
\left. \frac{\partial}{\partial r} \hat{\nu}^{S(r)} \right|_{r=a^+} = \left. \frac{\partial}{\partial r} \hat{\nu}^{S(r)} \right|_{r=a} - \frac{1}{6\pi \eta a^2} K^{\text{ext}}, \\
(3.13)
\]

where we have used the fact that the average of the induced force is proportional to the total external force acting on the drop, as given by Eq. (3.8), and that the surface average of the Oseen tensor is given by Eq. (A.6). Introducing these explicit expressions in Eq. (3.11), substituting it in (3.9) and using the constitutive relation (2.4), we arrive at
Introducing (3.14) and (3.8) in (3.4) one obtains an expression for the hydrodynamic force acting on the drop,

\[ f^{\text{hyd}} = \frac{2\eta^0}{1 + \alpha} \left\{ -(2 + 3\alpha)u + 5\alpha v^0_v + 2(1 - \alpha) \frac{\partial}{\partial a} \frac{\eta^0}{4\pi\eta^0} K^{\text{ext}} \right\} \]

(3.15)

In the case of a drop at rest in a quiescent fluid, this expression coincides with the result obtained by Subramanian [7], relating the capillary force to the average of the surface tension gradient. Note that to obtain this equation no assumption has been made about the origin of this gradient. We will evaluate this average in the next subsection in the case in which the surface tension depends only on temperature. The difference with the calculation in Ref. [7], lies in the fact that we will take into account the heat exchange at the interface given by Eq. (2.19).

### 3.2. The temperature averages

We will now proceed to the calculation of the average of the surface tension gradient appearing in Eq. (3.15). Since we consider that the surface tension only depends on the temperature, the gradient of \( \gamma \) is proportional to the temperature gradient.

Assuming that the derivative of the surface tension with respect to the temperature is constant, we have \( \nabla_s \gamma^s = \gamma_T \nabla_s T^s \). As has been done for the velocity field, we take the average of the formal solution of the temperature field, given by Eq. (2.31). Considering its surface gradient, and using Eq. (A.12) we get

\[ \nabla_s T^s = \nabla_s T^0_s + \int d'l' \nabla_s G(r - r')^s Q^{\text{ind}}(r') \]

\[ = \nabla_s T^0_s + \frac{2}{3\lambda^0} \eta q^{\text{ind}}^s. \]

(3.16)

which contains the surface dipole of the induced heat source. Taking into account Eq. (2.28), we get

\[ \eta q^{\text{ind}}^s = (1 - \beta^{-1}) \hat{\eta} \cdot \hat{v}^s - \beta^{-1} (e_s - \gamma) \hat{\eta} \nabla_s \cdot \hat{v}^s. \]

(3.17)

The first term on the right hand side of Eq. (3.17) is proportional to the heat flux arriving at the interface from the outer fluid. Using Eq. (2.14), the formal solution of the temperature field (2.31) and Eq. (A.11), we can write its surface average as
As regards the second term on the right hand side of Eq. (3.17), we will take into account the fact that we are only keeping terms up to linear order in the deviation from equilibrium. Then, we can substitute the local value of \((e_s - \gamma)\) by an average value,

\[
(e_s - \gamma)\hat{n} \nabla_s \cdot \mathbf{v}^S \approx (e_s - \gamma)\hat{n} \nabla_s \cdot \mathbf{v}^S.
\]

Moreover, we will use the incompressibility of the fluid, which allows us to express the surface divergence of the velocity as

\[
\nabla \cdot \mathbf{v} = \frac{\partial v_n}{\partial r} = \frac{\partial v_n}{\partial r}.
\]

Therefore, the average in Eq. (3.19) reduces to the calculation of the term

\[
-\hat{n} \frac{\partial v_n}{\partial r} = -\frac{\partial}{\partial r} \hat{n} \mathbf{v}^S = -\frac{1}{a} \left[ v_0^S - u + \frac{1}{6\pi\eta^0a} K_{\text{ext}} \right]
\]

where we have used the results of [17] for the averages of the velocity field.

Now substituting Eqs. (3.18) and (3.21) in Eq. (3.17) one obtains

\[
\hat{h} q_{\text{ind}}^S = (1 - \beta^{-1}) \left[ -\lambda^0 \hat{n} \nabla T_0^S + \frac{2}{3} \hat{h} q_{\text{ind}}^S \right]
\]

\[
+ \frac{e_s - \gamma}{a\beta} \left[ v_0^S - u + \frac{1}{6\pi\eta^0a} K_{\text{ext}} \right],
\]

which, solving for \(\hat{h} q_{\text{ind}}^S\) yields

\[
\hat{h} q_{\text{ind}}^S = \frac{1 - \beta}{2 + \beta} \lambda^0 \hat{n} \nabla T_0^S + \frac{3(e_s - \gamma)}{a(2 + \beta)} \left[ v_0^S - u + \frac{1}{6\pi\eta^0a} K_{\text{ext}} \right].
\]

This equation expresses the fact that the surface dipole moment of the induced heat source is a function of temperature gradient averages, related to differences in heat conductivity of both media, and averages of the velocity field, arising from the coupling between both fields through the inhomogeneities of the temperature-dependent surface tension. Insertion of (3.23) in (3.16) then gives the needed average of the surface tension gradient.

### 3.3. The Faxén theorem

Having computed the averages of the induced distributions, it is now possible to arrive at the expression of the Faxén theorem. In fact, using Eqs. (3.23), (3.16) and (3.15) one obtains
\[ K^{\text{hyd}} = \frac{2\pi\eta^o a}{1 + \alpha + \frac{2E}{3(\alpha + 2\beta + \gamma)}} \left\{ 5\alpha\bar{v}_0^V - 2 \left( \alpha - 1 - \frac{E}{\beta + 2} \right) \bar{v}_0^S + (\alpha - 1) a \frac{\partial}{\partial a} \bar{v}_0^S \right. \\
\left. - \left( 2 + 3\alpha + \frac{E}{\beta + 2} \right) u - \frac{a\gamma T}{\eta^o} \left[ \nabla_s T_0^S + 2(1 - \beta) \frac{\nabla \cdot \bar{v}_0^S}{\nabla T_0^S} \right] \right\}, \]  
(3.24)

where we have introduced the coefficient \( E = -\gamma T (\epsilon_\sigma - \gamma) / \lambda^o \eta^o \).

The Faxén theorem can also be expressed in terms of the values of the unperturbed fields and their gradients, evaluated at the center of the drop if one considers that for a velocity field satisfying the Stokes equations one has [3]

\[ \bar{v}_0^S = v_0(0) + \frac{a^2}{6} \Delta v_0(0), \]  
(3.25)

\[ \bar{v}_0^V = v_0(0) + \frac{a^2}{10} \Delta v_0(0), \]  
(3.26)

whereas for a temperature field satisfying the Laplace equation one can show that

\[ \nabla_s T_0^S = \frac{2}{3} \nabla T_0(0), \]  
(3.27)

\[ \nabla \cdot \bar{v}_0^S = \frac{1}{3} \nabla T_0(0). \]  
(3.28)

We can then rewrite Eq. (3.24) as

\[ K^{\text{hyd}} = \frac{2\pi\eta^o a}{1 + \alpha + \frac{2E}{3(\alpha + 2\beta + \gamma)}} \left[ 3\alpha + 2 + \frac{2E}{2 + \beta} \right] (-u + v_0(0)) \right] \\
+ \left( \alpha + \frac{2E}{3(2 + \beta)} \right) \frac{a^2}{2} \Delta v_0(0) + \frac{2a\gamma T}{\eta^o(2 + \beta)} \nabla T_0(0) \right]. \]  
(3.29)

Eqs. (3.24) or (3.29) then constitute the formulations of the Faxén theorem for a drop moving through a fluid under velocity and temperature gradients, and are the main results of this paper. For \( E = 0 \) we get the corresponding expression obtained by Subramanian [7]. Moreover, in the absence of external thermal gradients, we obtain an expression of the Faxén theorem which generalizes the one derived by Hetsroni et al. [5] in the absence of thermocapillary effects \( (E = 0) \).

If we compare Eq. (3.29) with Eq. (3.2), we can identify the mobility and the thermocapillary coefficient,

\[ \mu = \frac{1}{6\pi\eta^o a} \left( \frac{3 + 3\alpha + 2E}{2 + \beta + \gamma} \right), \]  
(3.30)

\[ \tau = -\frac{2a\gamma T}{\eta^o(2 + \beta)} \left( 3\alpha + 2 + \frac{2E}{2 + \beta} \right), \]  
(3.31)

which where implicitly present in Ref. [9]. Note that the mobility coefficient depends on \( E \), which means that the motion of a drop in a quiescent fluid will be affected by thermocapillarity even in the absence of external temperature gradients. In fact, the
motion of the drop will create local inhomogeneities in the temperature field which will cause a decrease of its velocity.

Experimental values of $E$ for bubbles moving in different liquids have been reported in [9]. Typically, $E$ is of order one for a wide variety of liquids, ranging from cryogenic liquids, as methane, to common liquids like water. Particularly, the value of $E$ for bubbles in water is higher than .7 for $T$ larger than 500 K, or in liquified CO2 for $T$ larger than 250 K.

4. Conclusions

In this paper we have presented a derivation of the Faxén theorem for a spherical drop or bubble moving through a fluid which itself is in steady motion and it is subject to a temperature gradient. It is interesting to realize that the velocity field depends on the temperature distribution since the surface tension depends on temperature. The temperature field, on the other hand, also depends on the velocity field because the inhomogeneities of the surface tension affect the heat exchange at the interface. Therefore, one should solve consistently both fields in order to arrive at the Faxén theorem when thermocapillary effects are present. Moreover, note that due to the structure of the hydrodynamic equations, this is the only way in which velocity and temperature can be coupled in the absence of convective contributions [19].

Our starting point was the formulation of the problem in terms of induced force and heat source densities. This method has previously been used in Ref. [12] in order to study the mobility of a suspension of drops in the absence of thermocapillary effects. In our case, these induced fields are determined in such a way that the boundary conditions on the surface of the drop are satisfied, enabling one to obtain a general formulation of the Faxén theorem in which the force exerted on the drop is expressed in terms of its velocity and averages of the velocity and temperature gradient of the unperturbed fluid. One advantage of this method is that we do not need to impose the form of the normal component of the velocity at the interface, as done in all previous calculations. Instead, only the value of the velocity of the center of mass of the drop is needed.

Expression (3.29) does not reduce to the sum of the contributions obtained by Hetsroni et al. [5] for a drop in a velocity gradient in thermal equilibrium and the one by Subramanian [7] for the motion of a drop in a quiescent fluid under a thermal gradient. The mobility and thermocapillary coefficients couple both contributions, since both depend on $E$, and therefore the coupling appears due to the nonlinear dependence on $\gamma T$. Note that to linear order, the global Faxén theorem is the sum of the two contributions, and only when thermocapillary effects are important, this additivity breaks down.

Thus, we have recovered several expressions that are valid in particular situations and which had been proposed previously by other authors. It is interesting to emphasize that in the derivation of the theorem we have not specified the nature of the fluid inside the drop and may consequently be applied to the case of bubbles for which the inner phase
is a gas.

Our analysis provides a systematic scheme to study the dynamics of the drop in a general situation. One may for instance use it to study the influence of the temperature dependence of the surface tension on Brownian motion.

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Appendix A. Calculation of Green’s function averages

In this appendix we derive the averages of the propagators which are needed to arrive at the main results presented in this paper. It is worth noting that the averages will be performed on spheres of radius $a$, because these averages appear in expressions which are already of first order in the imposed gradients. Since we are performing all the calculations to linear order in these gradients, departures from the spherical shape in these integrals will give rise to second order contributions and are therefore neglected.

First of all, we will compute the surface average of the Oseen tensor $H(r - r')$ with respect to the variable $r$, maintaining $r'$ fixed. This average is a second rank tensor, which will be of the form

$$\overline{H(r - r')}^S = \frac{1}{8\pi\eta^0} \left(A1 + B' r'\right).$$

since there is no preferred direction, and where $A$ and $B$ are functions of $r'$. Contraction of this equation with the unit tensor and with $r' r'$ gives the system of equations

$$3A + B = 4 \frac{1}{|r - r'|};$$

$$A + B = \frac{1}{|r - r'|} + \frac{(r \cdot r' - r')^2}{|r - r'|^3}.$$ 

These averages will be computed performing the integrals explicitly. Taking polar coordinates with the $z$-axis in the direction of $\hat{r}'$ gives

$$\frac{1}{|r - r'|} = \frac{1}{2} \int_{-1}^1 \frac{dx}{\sqrt{r^2 + r'^2 - 2rr'x}} = \begin{cases} 1/r, & r' < r; \\ 1/r', & r' > r, \end{cases}$$

(A.4)
\[
\frac{(r \cdot \hat{r}' - r')^2}{|r - r'|^3} = \frac{1}{2} \int_{-1}^{1} \frac{(rx - r')^2}{(r^2 + r'^2 - 2rr'x)^{3/2}} \, dx
\]

\[
= \begin{cases} 
1/3r, & r' < r; \\
1/3rr' - 3/2, & r' > r,
\end{cases}
\]  

(A.5)

where \( x \) stands for \( \cos \theta \). Using these explicit solutions we can solve the system of Eqs. (A.2)-(A.3) yielding

\[
\bar{H}(r - r')^S = \begin{cases} 
\frac{1}{6\pi \eta^0 r} \mathbf{1}, & r' < r; \\
\frac{1}{8\pi \eta^0 r'} \left[ 1 + \frac{2}{3} \hat{r}' \hat{r}' - \frac{r^2}{3r'^2} \right], & r' > r.
\end{cases}
\]  

(A.6)

The volume average of the Oseen tensor can be easily computed once its surface average is known. For the volume average one then obtains

\[
\bar{H}(r - r')^V = \frac{3}{4\pi a^3} \int_0^a \frac{dr}{4\pi r^2} \bar{H}(r - r')^S
\]

\[
= \frac{r'^2}{20\pi \eta^0 a^3} \left[ 31 + 5(\lambda^2 - 1) - 1 \right],
\]  

(A.7)

where we have only considered the situation when \( r' < r \).

When performing the temperature averages, we should also evaluate some surface averages of the propagator associated with the Laplace equation. Specifically, in Eq. (3.16) and (3.18) we have to determine the values of \( \bar{\nabla} \hat{G}(r - r')^S \) and \( \bar{\mathbf{\hat{h}}} \cdot \bar{\nabla} \hat{G}(r - r')^S \). Since the surface gradient is related to the gradient through Eq. (2.9), we should only calculate \( \bar{\nabla} \hat{G}(r - r')^S \) and \( \bar{\mathbf{\hat{h}}} \cdot \bar{\nabla} \hat{G}(r - r')^S \). We will first evaluate the term \( \bar{\nabla} \hat{G}(r - r')^S \). Using the symmetry of the propagator respect to \( r \) and \( r' \) one has

\[
\bar{\nabla} \hat{G}(r - r')^S = -\bar{\nabla}' \hat{G}(r - r')^S = -\frac{1}{4\pi \lambda^0} \hat{r}' \frac{1}{|r - r'|}
\]

\[
= \begin{cases} 
0, & r' < r; \\
\frac{r'}{4\pi \lambda^0 r'^3}, & r' > r,
\end{cases}
\]  

(A.8)

where the surface average has been calculated by means of the result (A.4). Regarding the average \( \bar{\mathbf{\hat{h}}} \cdot \bar{\nabla} \hat{G}(r - r')^S \), it should be a vector, so it has to be proportional to \( \hat{r}' \), and therefore it is expressed as

\[
\bar{\mathbf{\hat{h}}} \cdot \bar{\nabla} \hat{G}(r - r')^S = \frac{C}{4\pi \lambda^0} \hat{r}'.
\]  

(A.9)

where the scalar \( C \) can now be found by contraction of Eq. (A.9) with \( \hat{r}' \),

\[
C = (\hat{\mathbf{\hat{h}}} \cdot \hat{r}') (\hat{\mathbf{\hat{h}}} \cdot \bar{\nabla}) \frac{1}{|r - r'|} = \frac{\partial}{\partial r} \frac{\hat{\mathbf{\hat{h}}} \cdot \hat{r}'}{|r - r'|} = \begin{cases} 
-2r'/3r^3, & r' < r; \\
1/3r'^2, & r' > r.
\end{cases}
\]  

(A.10)
Substituting Eq. (A.10) in Eq. (A.8), and employing Eq. (2.9), we then obtain the surface averages

\[
\langle \hat{n} \hat{n} \cdot \nabla G(r - r') \rangle_s = \frac{2}{3} \frac{\hat{\rho}}{4\pi \lambda^0 a^2},
\]

(A.11)

\[
\nabla_s G(r - r') = \frac{2}{3} \frac{\hat{\rho}}{4\pi \lambda^0 a^2},
\]

(A.12)

where one should take into account that when performing these averages, the inequality \( r' < r \) must be considered.

References